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Cluster dynamics of a uniform chain of dissipatively coupled rotators☆

N.N. Verichev*, S.N. Verichev, V.I. Yerofeyev

Nizhnii Novgorod, Russia

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ABSTRACT

The existence and stability of stationary cluster structures in uniform chains of dissipatively coupled rotators is investigated. Cluster synchronization is interpreted as the classical synchronization of cluster rotators, which are elementary structure-forming objects. The complete set of types of cluster rotators and simple cells is defined. This definition is equivalent to the definition of the complete set of types of cluster structures. The completeness of this set is proved. The problem of the stability of cluster structures is solved. Physical examples of chains of rotators and a physical interpretation of the results of this research are given.

In uniform lattices a group of elements that have identical dynamical behaviour is called a cluster. In particular, if elementary systems are self-oscillating, the identical nature of their behaviour is specified by their synchronization. A cluster structure is an ordered ensemble of clusters that represents spatiotemporal order with respect to the natural coordinates of the lattice. The result of the formation of a cluster structure is called "cluster synchronization" and is interpreted from the point of view of the existence and stability of invariant manifolds in the lattice.[1–3]

The simple physical nature of cluster structures in lattices of oscillators was demonstrated in Ref. 4, and they were interpreted as a result of the simple (in the sense of a rotation number[5]) classical synchronization of the structure-forming objects. A uniform lattice itself is a special case of a coupled system of these objects.

Unlike the investigation of lattices of oscillators, the investigation of the dynamics of lattices of rotators involves additional difficulties due to the cylindrical nature of the phase space, which, in particular, complicates the effective use of Lyapunov function methodology. For this reason, the number of analytical studies of the cluster dynamics of lattices of rotators is small.[6,7]

As a continuation of previous work[4], the cluster structure in a very simple lattice, viz., a chain of rotators with Neumann boundary conditions, is investigated below.

1. General concepts and definitions

Definition 1. We define a rotator as a dynamical system of the form

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) \tag{1.1}$$

that is assigned in the cylindrical phase space

$$G(\mathbf{X}) = S^1 \times R^m, \quad \mathbf{X} = (\varphi, x_1, x_2, \ldots, x_m)^T, \quad \varphi = \varphi + 2\pi(\mathrm{mod}2\pi), \quad \varphi \in S^1, \quad x_i \in R^1$$

$$\mathbf{F}(\mathbf{X}): S^1 \times R^m \to S^1 \times R^m$$

System (1.1) can be non-autonomous. One of the variables, for example, x_m, can serve as "time." In the non-autonomous case, we will assume that the system is periodic in time.

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* Corresponding author.
E-mail addresses: s_veritchev@front.ru, erf04@sinn.ru (N.N. Verichev).

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The simplest rotator is described by the well-known second-order differential equation of a pendulum

$$\ddot{\varphi} + g(\varphi)\dot{\varphi} + f(\varphi) = \gamma$$

When different forms of the periodic functions $g(\varphi)$ and $f(\varphi)$ are used, this equation describes the dynamics of a physical pendulum moving under the action of a constant torque in a viscous medium, a superconducting Josephson junction,[8] a synchronous electrical device,[9] a phase synchronization system[10] and many other physical systems.

Henceforth we will call system (1.1) elementary, and we will assume that its dynamics are known.

Definition 2. We will define a symmetrical (asymmetrical) cluster rotator, i.e., a cluster rotator of type $R_s(n)$ (type $R_a(n)$), as a system of n coupled elementary rotators of the form

$$\dot{\mathbf{X}} = \mathbf{G}(\mathbf{X}) - \sigma\mathbf{B}_\chi \otimes \mathbf{C}\mathbf{X}$$

$$\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T, \quad \mathbf{G}(\mathbf{X}) = (\mathbf{F}(\mathbf{X}_1), \mathbf{F}(\mathbf{X}_2), \ldots, \mathbf{F}(\mathbf{X}_n))^T$$

$$\mathbf{B}_\chi = \begin{Vmatrix} 1 & -1 & 0 & \ldots & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & -1 & 0 \\ 0 & 0 & 0 & \ddots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -b_\chi & b_\chi \end{Vmatrix}, \quad \chi = s(\chi = a), \quad b_s = 1, \quad b_a = 2 \tag{1.2}$$

provided there is an attractor $A_s(n)$ ($A_a(n)$) such that

$$\forall(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \in A_s(n)\ (\in A_a(n)), \quad \mathbf{X}_i \neq \mathbf{X}_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n$$

where $\mathbf{C} = \mathrm{diag}(c_1, c_2, \ldots, c_{m+1})$ is the coupling matrix of the elementary rotators, in the phase space $T^n \times R^{mn}$ of the system. The values of the elements c_i are equal to zero or unity, σ is the scalar coupling parameter, and $\otimes$ is the symbol of a direct matrix product.

The names of the cluster rotators are associated with the symmetrical (asymmetrical) character of the "cluster" matrix $\mathbf{B}_\chi$. In the general case, a cluster attractor corresponds to the conditions of regular or chaotic stationary pulsations in the system of elementary rotators that make up a cluster rotator.

According to the definition of a cluster attractor, the attractor $A_\chi(n)$ lies outside of any invariant manifolds of system (1.2), and among the elements $R_\chi(n)$ there are no identically synchronized elements. A symbolic representation of a cluster rotator of type $R_s(n)$ in the "working" regime is shown in Fig. 1*a*. Each of the elementary rotators has its own shading. This reflects the fact that there are no mutual synchronizations.

Consider a system of k coupled cluster rotators of type $R_s(n)$ that have the form

$$\dot{\mathbf{X}} = \mathbf{G}(\mathbf{X}) - \sigma\mathbf{B}_s \otimes \mathbf{C}\mathbf{X} + \sigma_1\mathbf{C}_* \otimes \mathbf{C}(\mathbf{Y} - \mathbf{X})$$

$$\dot{\mathbf{X}} = \mathbf{G}(\mathbf{Y}) - \sigma\mathbf{B}_s \otimes \mathbf{C}\mathbf{Y} - \sigma_1\mathbf{C}_* \otimes \mathbf{C}(\mathbf{Y} - \mathbf{X}) + \sigma_2\mathbf{C}^* \otimes \mathbf{C}(\mathbf{Z} - \mathbf{Y})$$

$$\dot{\mathbf{Z}} = \mathbf{G}(\mathbf{Z}) - \sigma\mathbf{B}_s \otimes \mathbf{C}\mathbf{Z} - \sigma_2\mathbf{C}^* \otimes \mathbf{C}(\mathbf{Z} - \mathbf{Y}) + \sigma_3\mathbf{C}_* \otimes \mathbf{C}(\ldots - \mathbf{Z})$$

$$\vdots$$

$$\dot{\mathbf{W}} = \mathbf{G}(\mathbf{W}) - \sigma\mathbf{B}_s \otimes \mathbf{C}\mathbf{W} + \sigma_k\mathbf{C}_* \otimes \mathbf{C}(\ldots - \mathbf{W})$$

$$\mathbf{C}_* = \mathrm{diag}(0, 0, \ldots, 0, 1), \quad \mathbf{C}^* = \mathrm{diag}(1, 0, \ldots, 0, 0) \tag{1.3}$$

System (1.3) is written for an even number of rotators k. In the case of an odd value of k, the replacement $\mathbf{C}_* \to \mathbf{C}^*$ should be made in the last equation of the system.

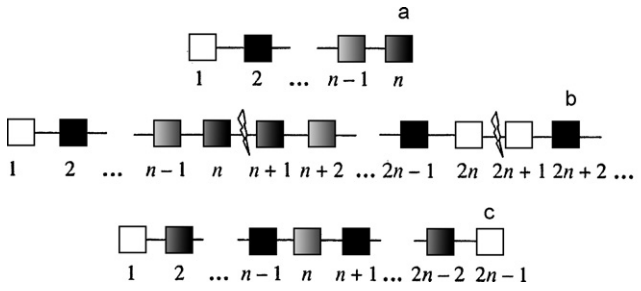

Fig. 1.

We note the obvious. System (1.3) has a "principal" integral manifold $M(n) = \{\mathbf{X} = \mathbf{Y} = \mathbf{Z} = \ldots = \mathbf{W}\}$. It is also obvious that this manifold is filled with trajectories of a cluster rotator of type $R_s(n)$. The cluster attractor (attractors) $A_s(n) \subset M(n)$. Thus, if the entire manifold or the part of it that contains $A_s(n)$ together with its region of attraction[11] is stable for certain σ_i, a regime of simple synchronization of the cluster rotators on the attractor $A_s(n)$ is also stable.

We now assume that

$$\sigma_1 = \sigma_2 = \ldots = \sigma, \quad \mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T, \quad \mathbf{Y} = (\mathbf{X}_{2n}, \mathbf{X}_{2n-1}, \ldots, \mathbf{X}_{n+1})^T,$$
$$\mathbf{Z} = (\mathbf{X}_{2n+1}, \mathbf{X}_{2n+2}, \ldots, \mathbf{X}_{3n})^T \ldots$$

Under this assumption, when system (1.3) is written relative to elementary rotators, it represents a uniform chain of the form

$$\dot{\mathbf{X}}_i = \mathbf{F}(\mathbf{X}_i) + \sigma\mathbf{C}(\mathbf{X}_{i-1} - 2\mathbf{X}_i + \mathbf{X}_{i+1}); \quad \mathbf{X}_0 \equiv \mathbf{X}_1, \quad \mathbf{X}_N \equiv \mathbf{X}_{N+1}; \quad i = 1, 2, \ldots, N,$$
$$N = nk \tag{1.4}$$

Thus, on the one hand, the chain of rotators (1.4) is a special case of the system of cluster rotators (1.3). On the other hand, if there is a regime with simple synchronization of the cluster rotators in system (1.3), there will be a cluster structure with the sequence of clusters depicted in Fig. 1*b* in the corresponding chain.

If the number of cluster rotators is even, the chain will have the so-called "central" cluster structure, and if the number is odd, the chain will have the "alternative" structure.[2,3]

We will assume that the synchronization regime is perfectly exact (the non-physical character of the situation is of no significance). In such a case, the connections between the synchronized rotators can be broken without disrupting the dynamical regime. In electrical circuits such a procedure corresponds to cutting connections that join equipotential points (in Fig. 1*b* a symbolic knife is drawn on the connections between the synchronized rotators). The "partitionability" of the cluster structure into an even number of identical structure-forming objects is one of its main properties.

The cluster rotators of type $R_a(n)$ differ from the cluster rotators of type $R_s(n)$ not only in their matrices. They also exist only in pairs and are joined to one another with degeneracy of the number of degrees of freedom by the number of degrees of freedom of one elementary rotator (rigid coupling). The synchronization regime of a coupled pair forms a non-partitionable cluster structure.

Definition 3. The union of identical cluster rotators, whose synchronization regime defines a non-partitionable cluster structure, is called a simple cell.

By definition, a cluster rotator of type $R_s(n)$ is itself a simple cell. A simple cell based on a pair of cluster rotators of type $R_a(n)$ is shown in Fig. 1*c*. The middle elementary oscillator (the *n*th elementary oscillatory) is "shared."[4] The elementary rotators that are equidistant from the ends are synchronized rotators (the central structure). The imposition of rigid connections between synchronized oscillators (between their corresponding variables) clearly does not disrupt the dynamical regime of the simple cell. In electrical circuits this corresponds to short-circuiting equipotential points. In the symbolic representation this procedure corresponds to folding of the simple cell at the central oscillator with the superposition of squares that have the same shading, i.e., convolution. As a result of this folding, the cell is reduced to one cluster rotator, i.e., $R_a(n)$. "Convolutability" to one cluster rotator is a principal property of a simple cell. Note that the dynamical system of the simple cell consists of Eq. (1.2) with a $(2n-1)\times(2n-1)$ matrix. The synthesis of a cluster structure based on synchronized simple cells occurs according to the same principle as in the case of the cluster rotators $R_s(n)$. In Fig. 1*b* $R_s(n)$ is replaced by the simple cell $R_a(n)$.

1.1. General properties of cluster structures

1°. Any stationary cluster structure in a chain corresponds uniquely to the simple synchronization of a certain number of identical cluster rotators of one of two types: $R_s(n)$ or $R_a(n)$.
2°. Any cluster structure can be partitioned into an integer number of identical simple cells, which are convolutable to a one cluster rotator.
3°. A structure that does not have the partitionability and convolutability properties, i.e., is not reducible to one cluster rotator, is not a cluster structure and does not exist.
4°. Let p be the number of all odd cofactors of the number N, and let q be the number of all of its even cofactors (simple and composite cofactors, cofactors that differ from unity and the number N itself). If N is an odd number, p alternative cluster structures based on cluster rotators of type $R_s(n)$ and $p+1$ central cluster structures based on cluster rotators of type $R_a(n)$ exist in a chain. If N is an even number, p alternative cluster structures based on cluster rotators of type $R_s(n)$, p central cluster structures based on cluster rotators of type $R_a(n)$ and q central cluster structures based on cluster rotators of type $R_s(n)$ exist. The total number of cluster structures, including the single-cluster (trivial) and N-cluster structures is equal to $2p+3$ if N is odd and to $2p+q+2$ if N is even.

2. The model

Consider a dynamical system of the form

$$C\ddot{\varphi}_i + \dot{\varphi}_i + \sin\varphi_i = \gamma + \sigma(\dot{\varphi}_{i-1} - 2\dot{\varphi}_i + \dot{\varphi}_{i+1}) + A\sin\psi, \quad \dot{\psi} = \omega_0, \quad i = 1, 2, \ldots, N \tag{2.1}$$

which is assigned in the toroidal phase space

$$G(\varphi_1, \ldots, \varphi_N, \psi, \dot{\varphi}_1, \ldots, \dot{\varphi}_N) = T^{N+1} \times R^N$$

where $C > 0$ and $\gamma > 0$ are dimensionless parameters that have the meaning of the moment of inertia and constant angular momentum of the rotator, A and ω_0 are the amplitude and frequency of the external action, and σ is the coupling parameter of the rotators.

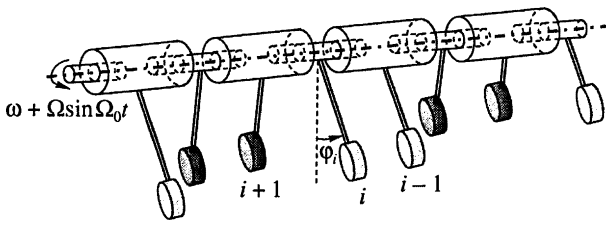

Fig. 2.

The system is considered with the Neumann boundary conditions

$$\varphi_0 = \varphi_1, \quad \varphi_N = \varphi_{N+1} \tag{2.2}$$

Equations of type (1.1) describe the dynamics of a whole class of mechanical, electromechanical, quantum-mechanical and other systems.

Example 1. Fig. 2 depicts a chain of Froude pendulums.[12] The suspending sleeves of the pendulums are slipped over a shaft, which is brought into rotational motion with constant frequency ω or into rotation-oscillatory motion with frequency $\omega + \Omega \sin \Omega_0 t$. There is viscous friction between all the sleeves and the shaft, as well as between all the contacting sleeves. The walls of the outer sleeves are fairly thin at the points of contact, so that the lengths of the pendulums are assumed to be identical.

In dimensional variables and parameters, the equations of motion of the system, i.e., the chain of pendulums, have the form

$$ml^2\ddot{\varphi}_i + d\dot{\varphi}_i + mgl\sin\varphi_i = R(\omega + \Omega\sin\psi - \dot{\varphi}_i) + \lambda(\dot{\varphi}_{i-1} - \dot{\varphi}_i) + \lambda(\dot{\varphi}_{i+1} - \dot{\varphi}_i),$$
$$\dot{\psi} = \Omega_0, \quad i = 1, 2, \ldots, N$$

with boundary conditions (2.2).

Here $R(\omega + \Omega\sin\psi - \varphi_i)$ is the moment of the viscous friction force created by the shaft, and $\lambda(\dot{\varphi}_{i-1} - \dot{\varphi}_i)$ and $\lambda(\dot{\varphi}_{i+1} - \dot{\varphi}_i)$ are the moments of the friction forces created by adjacent pendulums. After dividing the equations by *mgl* and changing to dimensionless time, we obtain Eq. (2.1). The relation between the dimensionless time and the dimensional time, and the relations between the dimensionless parameters and the dimensional parameters of the model have the forms

$$\tau = \frac{mgl}{d+R}t, \quad C = \frac{l}{g}\left(\frac{mgl}{d+R}\right)^2, \quad \gamma = \frac{R\omega}{mgl}, \quad \sigma = \frac{\lambda}{d+R}, \quad \omega_0 = \Omega_0\frac{d+R}{mgl}$$

Example 2. The state of a superconducting junction (a Josephson contact) is described by two variables: the phase difference φ between the quantum-mechanical order functions and the potential difference between the superconductors $\dot{\varphi} = V$ (in dimensionless time).[13] In the resistive model with dimensionless variables and parameters, a separate non-autonomous superconducting junction is described by the pendulum equation from system (2.1). The parameters of the model have the following physical meanings: *C* is the capacitance of the junction, γ is the current from the external source, and *A* and ω_0 are the amplitude and frequency of the external microwave field. In dimensionless form, $\dot{\varphi} = V = \omega$ is the Josephson relation.

Fig. 3 shows a chain of dissipatively coupled superconducting junctions (they are represented by crosses) with uniform injection of a constant current, which is immersed in an external microwave field.

For the current of an arbitrary junction, we have system (2.1), where σ is the conductivity (the reciprocal of the dimensionless resistance *R* of the junction). The condition $\langle\dot{\varphi}_i\rangle_t = 0$ (equilibrium states and oscillatory motions of the *i*th pendulum, Fig. 2) corresponds to the superconducting state of the junction, and the condition $\langle\dot{\varphi}_i\rangle_t \neq 0$ (rotational motions of the pendulum) corresponds to the resistive state (the generation regime).

From a comparison of the physical systems in Examples 1 and 2, we obtain the following equivalence between the electrical and mechanical quantities: *V* (the voltage) ~ $\dot{\varphi} = \omega$, 1/*R* (the conductivity of the junction) ~ σ (the coefficient of viscous friction), and *I* (the current) ~ *M* (the moment). In this case, Ohm's law $(V_{k+1} - V_k)/R = I_k$ expresses the moment of the viscous friction force exerted by the $(k+1)$th rotator on the *k*th rotator.

Theorem 1. *The cluster rotators $R_s(n)$ and $R_a(n)$ comprise the complete set of types of cluster rotators in a chain of elementary rotators with Neumann boundary conditions. Simple synchronization of these cluster rotators defines all possible types of cluster structures that exist in the chain.*

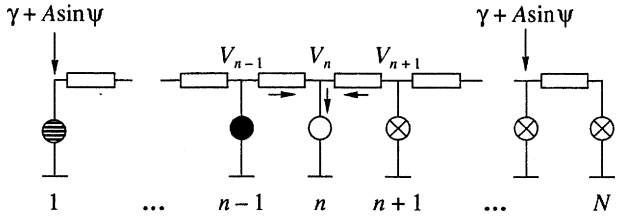

Fig. 3.

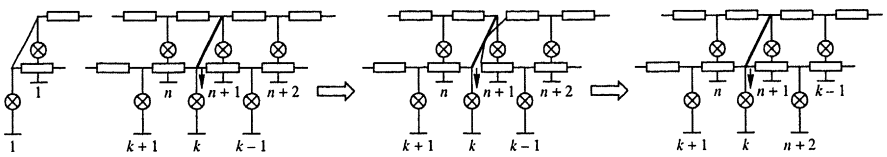

Fig. 4.

Proof. We will prove the theorem using elementary physics by considering an electronic circuit (Fig. 3) and taking into account the equivalence of the electrical and mechanical quantities. We will use "equipotential" transformations of the chain, i.e., removal (severing) of the connections that join equipotential points and the short-circuiting of such points. After these transformations, the dynamical regimes of the system under consideration remain unchanged.

We will assume that a certain cluster structure forms in the chain and that only the first n elementary rotators are not synchronized, i.e., the $(n+1)$th elementary rotator is synchronized with one of the first n elementary rotators.

We will initially assume that the $(n+1)$th rotator is synchronized with the kth rotator, provided that $k \le n-2$. In such a case the values of all the corresponding physical variables of the synchronized rotators are identical at any time. Their "input" points are equipotential, i.e., $V_k = V_{n+1}$ (Fig. 3). By using a contact bar to connect the equipotential points, we can perform the schematic transformations depicted in Fig. 4. From the equations for the currents in the synchronized rotators in the first and last positions of the diagram, we obtain

$$V_n - 2V_{n+1} + V_{n+2} = IR, \quad V_{k+1} - 2V_k + V_{n+2} = IR \Rightarrow V_n = V_{k+1}$$

There is a contradiction: the system consisting of the first n elementary rotators contains synchronized rotators. Thus, the $(n+1)$th rotator cannot be synchronized with any rotator with index k if $k \le n-2$. Two cases remain: either $k = n$ or $k = n-1$.

Let us assume that $k = n$, i.e., $V_n = V_{n+1}$, For the input currents of the synchronized rotators, we have

$$V_{n-1} - V = IR, \quad V_{n+2} - V = IR \Rightarrow V_{n-1} = V_{n+2}$$

i.e., the $(n-1)$th and $(n+2)$th rotators are also synchronized with one another. Writing the equations for the currents in these rotators, we obtain

$$V_{n-2} + V - 2V_{n-1} = I_{n-1}R, \quad V_{n+3} + V - 2V_{n+2} = I_{n+2}R \Rightarrow V_{n-2} = V_{n+3}$$

Continuing the sequence of equations for the currents in the synchronized rotators, we obtain $V_1 = V_{2n}$. The rotator numbered $2n+1$ can be synchronized either with the $2n$th rotator or with the $(2n-1)$th rotator. We will assume that it is synchronized with the $(2n-1)$th rotator, i.e., $V_{2n+1} = V_{2n-1}$. Writing down the equations for the currents in the first and $2n$th rotators, we obtain

$$V_2 - V_1 = IR, \quad V_{2n-1} + V_{2n} - 2V_{2n+1} = IR \Rightarrow V_2 = V_1$$

There is a contradiction: the system consisting of the first n rotators contains synchronized rotators. For example, the $(2n+1)$th rotator is synchronized with the $2n$th rotator. By repeatedly applying this reasoning, we obtain the sequence of synchronized rotators in the chain shown in Fig. 1*b*. We will show that in this case the cluster structure contains an integer number of cluster rotators of type $R_s(n)$, $N = mn$. Let us assume the opposite: in the sequence of synchronized rotators shown in Fig. 1*b*, the number of unsynchronized rotators at the end of the chain is equal to $k < n$. We remove all the cluster rotators from the structure, leaving the penultimate cluster rotator and the unsynchronized rotators just mentioned. We number the remaining rotators from unity to $k+n$. In this case we have the structure shown in Fig. 1*b* which ends with the rotator with the index $n+k$, i.e., $V_{n+k+1-i} = V_{n-k+i}$ $(i = 1, 2, \ldots k)$. The equations for the currents in the synchronized last rotator and the $(n-k+1)$th rotator have the form

$$V_{n+k-1} - V_{n+k} = RI, \quad V_{n-k} - 2V_{n-k+1} + V_{n-k+2} = RI, \quad V_{n+k} = V_{n-k+1},$$
$$V_{n+k-1} = V_{n-k+2}$$

From these equations we obtain that, $V_{n-k} = V_{n-k+1}$. If $k \neq n$, the system consisting of the first n rotators is not a cluster rotator, which contradicts the condition that the first n rotators are not synchronized, and if $k = n$, we have $V_0 = V_1$, which is the boundary condition.

Thus, under the condition $\mathbf{X}_n = \mathbf{X}_{n+1}$ a cluster structure based on cluster rotators of type $R_s(n)$ is formed. Taking into account this equality in the system, we obtain a subsystem consisting of the first n equations of type (1.2) with the matrix $\mathbf{B}_s$, i.e., a cluster rotator of type $R_s(n)$.

Finally, we will assume that the $(n+1)$th rotator is synchronized with the $(n-1)$th rotator, i.e., $V_{n-1} = V_{n+1}$. Writing the equations for the currents in these rotators, we obtain

$$V_{n-2} - 2V_{n-1} + V_n = IR, \quad V_{n+2} - 2V_{n+1} + V_n = IR \Rightarrow V_{n-2} = V_{n+2}$$

Successively writing the equations for the currents, we obtain $V_i = V_{2n-i}$, $(i = 1, 2, \ldots, n-1)$, i.e., a system of $2n-1$ rotators, which is a simple cell formed by a pair of cluster rotators of type $R_a(n)$. Taking into account the equality $\mathbf{X}_{n-1} = \mathbf{X}_{n+1}$ in system (1.4), we obtain a subsystem consisting of n first equations of the form (1.2) with the matrix $\mathbf{B}_a$, i.e., a cluster rotator of type $R_a(n)$. The proof that there is an integer number of simple cells in the cluster structure is carried out in the same way as above.

Thus, $R_s(n)$ and $R_a(n)$ make up the complete set of types of structure-forming objects in a chain with Neumann boundary conditions.

3. Autonomous chain

Setting $A = 0$ in equalities (2.1) and transforming the time by setting $\sqrt{C^{-1}}t = \tau$, we obtain the system

$$\ddot{\varphi}_i + \lambda\dot{\varphi}_i + \sin\varphi_i = \gamma + \sigma_0(\dot{\varphi}_{i-1} - 2\dot{\varphi}_i + \dot{\varphi}_{i+1}), \quad i = 1, 2, \ldots, N \tag{3.1}$$

with boundary conditions (2.2), where $\lambda = \sqrt{C^{-1}}$, $\sigma_0 = \sigma\sqrt{C^{-1}}$.

3.1. Some general properties of system (3.1)

1°. In the phase space $T^N \times R^N$ of the system, there are no closed trajectories of the oscillatory type.

In fact, the derivative of the periodic Lyapunov function

$$V = \sum_{i=1}^{N}\left(\frac{1}{2}\dot{\varphi}_i^2 + \int_{\varphi_{0i}}^{\varphi_i}(\sin\varphi_i - \gamma)d\varphi_i\right)$$

calculated by virtue of equalities (3.1) is non-positive over the entire phase space of the system:

$$\dot{V} = -\sum_{i=1}^{N}(\lambda\dot{\varphi}_i^2 + \sigma_0(\dot{\varphi}_i - \dot{\varphi}_{i+1})^2) \le 0$$

which demonstrates property 1°.

Thus, the limit sets of trajectories of system (3.1) are equilibrium states and (or) sets of the rotational type (rotationally-oscillatory and rotational limit cycles, tori and possibly chaotic limit sets).

2°. The phase space of the system contains the "principal" integral manifold

$$M(1) = \{\varphi_1 = \varphi_2 = \ldots = \varphi_N, \dot{\varphi}_1 = \dot{\varphi}_2 = \ldots = \dot{\varphi}_N\} \tag{3.2}$$

which has the form of a cylinder filled with trajectories of an elementary rotator of the form

$$\ddot{\varphi} + \lambda\dot{\varphi} + \sin\varphi = \gamma \tag{3.3}$$

This property is obvious.

The partitioning of the (γ, λ) parameter plane of an elementary rotator into regions that correspond to qualitatively different structures of the trajectories on the $(\varphi, \dot{\varphi})$ cylinder is well known.[9]

3.2. Brief information on the properties of Eq. (3.3)

Fig. 5 shows a bifurcation diagram of the (γ, λ) parameter plane (Fig. 5*a*) and phase portraits of the rotator on a scan of the $(\varphi, \dot{\varphi})$ phase cylinder for parameters from each region (Fig. 5*b*-*d*). Region *1* corresponds to global stability of the equilibrium state $O_1(\arcsin\gamma, 0)$ (Fig. 5*b*). The Tricomi curve $\lambda^*(\gamma)$ corresponds to a separatrix loop of the saddle $O_2(\pi - \arcsin\gamma, 0)$ (saddle node for $\lambda > \hat{\lambda}$), from which a stable limit cycle of the second kind is generated when the parameters transfer from region *1* into region *2* (when they transfer from region *1* to region *3*) (Fig. 5*c*). The straight line $\gamma = 1$ corresponds to a complex equilibrium state of the saddle-node type, which vanishes when the parameters transfer into region *3*. For region *3* there is a single, globally asymptotically stable limit cycle of the second kind (Fig. 5*d*).

3°. The stationary, spatially homogeneous dynamical regimes of the chain correspond to stable limit sets of phase trajectories that lie on the principal manifold (3.2). The stable equilibrium state $O_1(\arcsin\gamma, 0)$ corresponds to resting Froude pendulums suspended at the same angle; the limit cycle corresponds to in-phase rotation of all the pendulums in the chain. We will show that these limit sets are stable not only on the manifold, but also along directions that are transverse to manifold (3.2). The replacement of variables $u_i = \varphi_i - \varphi_{i+1}$, $\dot{u}_i = v_i$ transforms equalities (3.1) into manifold (3.2). Using Lagrange's theorem, we obtain

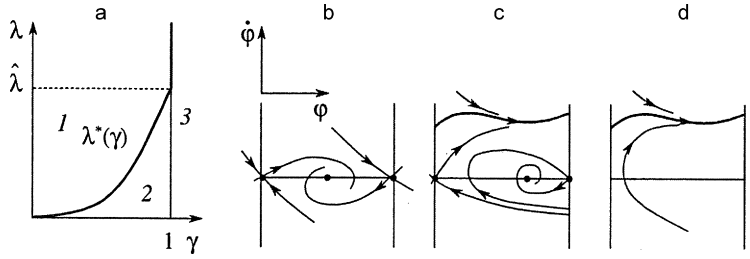

Fig. 5.

$$\begin{aligned}&\dot{u}_i = v_i\\&\dot{v}_i = -\lambda v_i - (\cos\xi_i)u_i + \sigma_0(v_{i-1} - 2v_i + v_{i+1}), \quad \xi_i \in (\varphi_i, \varphi_{i+1}), \quad i = 1, \ldots, N;\\&v_0 = 0, \quad v_N = 0\end{aligned} \tag{3.4}$$

3.3. Comments on system (3.4).

1°. The variables u_i, v_i ($i = 1, \ldots, N$) of system (3.4) are transverse to manifold (3.2), and the manifold itself corresponds to its trivial solution.

2°. If $\xi_1 = \varphi_1 = \varphi_2 = \xi_2 = \ldots = \varphi_N = \varphi$, where $\varphi(t)$ is an arbitrary solution and $(\varphi, \dot{\varphi}) \in M(1)$, system (3.4) is a system in variations relative to the entire manifold. If the solution $\varphi(t) = \varphi^*(t)$ corresponds to a stable limit set of the manifold, system (3.4) is a system in variations relative to the part of the manifold $M(1)$ that contains this limit set in its region of attraction.[11] Thus, if the trivial solution of system (3.4), written with respect to $\xi(t) = \varphi^*(t)$, is stable, the corresponding dynamical regime of the chain is also stable.

We will write system (3.4) in the form of one equation

$$\dot{\mathbf{U}} = (\mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi) + \sigma_0 \mathbf{D}_{N-1} \otimes \mathbf{C})\mathbf{U}, \quad \mathbf{U} = (u_1, v_1, u_2, v_2, \ldots, u_{N-1}, v_{N-1})^T$$

$$\mathbf{J}_0 = \begin{Vmatrix} 0 & 1 \\ -\cos\xi & -\lambda \end{Vmatrix}, \quad \mathbf{D}_{N-1} = \begin{Vmatrix} -2 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & -2 & 1 & \ddots & \ldots & 0 & 0 \\ 0 & 1 & -2 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{Vmatrix}, \quad \mathbf{C} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} \tag{3.5}$$

To prove the stability of the solution $\mathbf{U} = 0$, we present the following theorem.

Theorem 2. *The non-degenerate transformation* $\mathbf{U} = \mathbf{SV}$, *where* $\mathbf{S} = \mathbf{S}_0 \otimes \mathbf{I}_2$ *and* $\mathbf{S}_0$ *is the transforming matrix* $\mathbf{D}_{N-1}$, *transforms* Eq. (3.5) *into the equivalent equation*

$$\dot{\mathbf{V}} = (\mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi) - \sigma_0 \mathbf{D}_0 \otimes \mathbf{C})\mathbf{V}; \quad \mathbf{D}_0 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{N-1}) \tag{3.6}$$

where the $\lambda_j = 4\sin^2 \pi j/(2N)$ ($j = 1, 2, \ldots, N-1$) are *roots of the matrix* $\mathbf{D}_{N-1}$.

Proof. Making the replacement $\mathbf{U} = \mathbf{SV}$ in Eq. (3.5), we obtain the equation

$$\dot{\mathbf{V}} = (\mathbf{S}^{-1}(\mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi))\mathbf{S} - \sigma_0 \mathbf{S}^{-1}(\mathbf{D}_{N-1} \otimes \mathbf{C})\mathbf{S})\mathbf{V}$$

Using the properties of the direct product of matrices,[14] we have

$$\mathbf{S}^{-1}(\mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi))\mathbf{S} = (\mathbf{S}_0^{-1} \otimes \mathbf{I}_2)(\mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi))(\mathbf{S}_0 \otimes \mathbf{I}_2) =$$

$$= (\mathbf{S}_0^{-1}\mathbf{I}_{N-1} \otimes \mathbf{I}_2\mathbf{J}_0(\xi))(\mathbf{S}_0 \otimes \mathbf{I}_2) = \mathbf{S}_0^{-1}\mathbf{I}_{N-1}\mathbf{S}_0 \otimes \mathbf{I}_2\mathbf{J}_0(\xi)\mathbf{I}_2 = \mathbf{I}_{N-1} \otimes \mathbf{J}_0(\xi)$$

$$\mathbf{S}^{-1}(\mathbf{D}_{N-1} \otimes \mathbf{C})\mathbf{S} = (\mathbf{S}_0^{-1} \otimes \mathbf{I}_2)(\mathbf{D}_{N-1} \otimes \mathbf{C})(\mathbf{S}_0 \otimes \mathbf{I}_2) = \mathbf{S}_0^{-1}\mathbf{D}_{N-1}\mathbf{S}_0 \otimes \mathbf{I}_2\mathbf{C}\mathbf{I}_2 = \mathbf{D}_0 \otimes \mathbf{C}$$

Eq. (3.6) follows as a result.

Note that Eq. (3.6) decomposes into the $N-1$ second-order systems relative to $(x_i, y_i)^T = \mathbf{V}_i$:

$$\dot{x}_i = y_i, \quad \dot{y}_i = -\lambda y_i - (\cos\xi_i)x_i - \sigma_0\lambda_i y_i; \quad i = 1, 2, \ldots, N-1$$

If $\sigma_0 = 0$, each of these systems is a stable system in variations (an unperturbed system) relative to a solution corresponding to one of the limit sets of trajectories of the pendulum equation (a stable equilibrium state, a stable limit cycle). If $\sigma_0 \neq 0$, then for any fixed $t = t_0$ there is a right-handed rotation of the vector field of each of the systems on trajectories of the unperturbed system. Since all the trajectories of the unperturbed system (in the expanded phase space) enter into a certain cylinder $Z\{t > t_0, x^2 + y^2 \le R^2(t_0)\}$, $\lim\limits_{t_0 \to \infty} R(t_0) = 0$, and they do not leave it, this also applies to the trajectories of the perturbed system. Then, the stability of each of the systems indicated follows from the stability of the unperturbed system. Physically, this fact is trivial: additional dissipation is introduced into the stable system, and, naturally, it only improves its stability. Hence, the solution $\mathbf{V} = 0$ and, therefore, the solution $\mathbf{U} = 0$ are stable.

The spatially homogeneous dynamical regime of the chain is stable for any, including an arbitrarily low, level of diffusional coupling. In parameter region *1* (see Fig. 5*a*), when any initial conditions are assigned on the manifold $M(1)$ and in a small neighbourhood near it, the spatially homogeneous state of the chain is an equilibrium state. In parameter region *2*, either an equilibrium state or a spatially homogeneous regime of in-phase rotations of all the pendulums in the chain can be realized, depending on the initial conditions on the manifold $M(1)$. Finally, in parameter region *3*, a spatially homogeneous regime of in-phase rotations of all the pendulums in the chain is obtained under any initial conditions on the manifold $M(1)$ and in a small neighbourhood near it.

4. Cluster dynamics of an autonomous chain

4.1. Symmetric cluster rotators and cluster structures associated with them

We will introduce vectors and vector functions for an elementary rotator and a cluster rotator of type $R_s(n)$ in the following manner

$$\mathbf{X}_i = (\varphi_i, x_i)^T, \quad \mathbf{F}(\mathbf{X}_i) = (x_i, -\lambda x_i - \sin\varphi_i + \gamma)^T$$
$$\mathbf{X} = (\varphi_1, x_1, \ldots, \varphi_n, x_n), \quad \mathbf{G}(\mathbf{X}) = (x_1, -\lambda x_1 - \sin\varphi_1 + \gamma, \ldots, x_n, -\lambda x_n - \sin\varphi_n + \gamma)$$

As a result, we obtain equations of types (1.1) and (1.2), respectively.

An analytical investigation of the conditions for the existence of a cluster attractor (attractors) is a very complex problem in the general case. However, in special cases, for example, in the case of the parameters from region *2* (see Fig. 5*a*) and small values of the coupling parameter, their existence is almost obvious. In fact, let $n=2$ and let the parameters of the rotators belong to region *2*. When $\sigma_0=0$, there is a cluster attractor of the form $L_1 \times O_1^1$, i.e., the Cartesian product of a stable cycle of the first rotator and a stable equilibrium state of the second rotator. According to the continuity of the solutions with respect to σ_0, this limit set maintains topological equivalence and stability for sufficiently small values of σ_0.

Consider a system of k interacting cluster rotators of type $R_s(n)$ (1.3) under the condition $\sigma_{01}=\sigma_{02}=\ldots=\sigma_0^*$. Just as in Section 3, by replacement of variables

$$\mathbf{U}_1 = \mathbf{X}-\mathbf{Y}, \quad \mathbf{U}_2 = \mathbf{Y}-\mathbf{Z}, \quad \mathbf{U}_3 = \mathbf{Z}-\ldots, \ldots, \mathbf{U}_k = \ldots - \mathbf{W}$$

we transform the system into the manifold $M(n)=\{\mathbf{X}=\mathbf{Y}=\mathbf{Z}=\ldots=\mathbf{W}\}$ and write the system obtained in the form of the single equation

$$\dot{\mathbf{U}} = (\mathbf{I}_{k-1}\otimes\mathbf{J}_{2n}(\xi)+\sigma_0^*\mathbf{D}_{n(k-1)}\otimes\mathbf{C})\mathbf{U}, \quad \mathbf{U}=(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_{k-1})^T$$
$$\mathbf{J}_{2n} = \mathbf{G}'(\xi)-\sigma_0\mathbf{B}_s\otimes\mathbf{C}, \quad \xi\in A_s(n),$$

$$\mathbf{D}_{n(k-1)} = \begin{Vmatrix} -2\mathbf{C}_* & \mathbf{C}^* & 0 & \ldots & \ldots & 0 & 0 \\ \mathbf{C}_* & -2\mathbf{C}_* & \mathbf{C}_* & \ddots & \ldots & 0 & 0 \\ 0 & \mathbf{C}^* & -2\mathbf{C}_* & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2\mathbf{C}_* & \mathbf{C}_* & 0 \\ 0 & 0 & 0 & \ddots & \mathbf{C}_* & -2\mathbf{C}^* & \mathbf{C}_* \\ 0 & 0 & 0 & \ldots & \mathbf{0} & \mathbf{C}^* & -2\mathbf{C}_* \end{Vmatrix} \tag{4.1}$$

Here $\mathbf{J}_{2n}$ is the Jacobian matrix of the cluster rotator.

As above, the stability of a cluster state corresponds to the stability of the trivial solution of Eq. (4.1).

1°. The case k=2. Central n-cluster structure, N=2n. In this case, Eq. (4.1) takes the form

$$\dot{\mathbf{U}}_1 = (\mathbf{J}_{2n}(\xi)-\sigma_0^*\mathbf{D}_n\otimes\mathbf{C})\mathbf{U}_1; \quad \mathbf{D}_n = \mathrm{diag}(0,0,\ldots,0,2) \tag{4.2}$$

We will use the perturbation theory of Lyapunov exponents of the solutions of linear systems.[14–16] We will assume that the coupling between the cluster rotators, as well as the coupling between the elementary rotators of the cluster rotator, are fairly small. We will also assume that the cluster attractors are regular.

When $\sigma_0^*=0$, Eq. (4.2) is an equation in variations relative to the solution $\xi\in A_s(n)$. If the cluster attractor $A_s(n)$ is regular (a stable equilibrium state, a limit cycle or a torus), this ("generating") equation is stable. For the perturbation norm we have

$$\|\sigma_0^*\mathbf{D}_n\otimes\mathbf{C}\mathbf{U}\| \le \sigma_0^*\|\mathbf{D}_n\||\mathbf{U}| = \sigma_0^* D|\mathbf{U}|, \quad D = \|\mathbf{D}_n\|$$

In the present case $D=2$. By virtue of the linearity of the system, the estimate of the growth of its solutions is uniform with respect to the initial condition.[16] Hence we obtain an estimate for the leading exponent of the perturbed system

$$\chi(\mathbf{U}) \le \Lambda + D\sigma_0^*,$$

where $\Lambda<0$ is the leading exponent of the unperturbed equation. The estimate of the solutions of the perturbed equation has the form

$$|\mathbf{U}| \le |\mathbf{U}(t_0)| D\exp(\Lambda + D\sigma_0^* + \varepsilon)t$$

where the quantity $\varepsilon>0$ is as small as desired. If the parameter σ_0^* is so small that $\Lambda + D\sigma_0^* < 0$, then $|\mathbf{U}|\to 0$ as $t\to\infty$, i.e., the solution $\mathbf{U}=0$ of Eq. (4.2) and the cluster structure corresponding to it are stable. Setting $\sigma_0^*=\sigma_0$ (changing to a uniform chain), we obtain the stable cluster structure depicted in Fig. 1*b* if all the cluster rotators except the first two are removed from it. In a chain of Froude pendulums with $k=2$ and $n=2$, we obtain a two-cluster structure by removing two cluster rotators (four elementary rotators) shown in Fig. 2.

2°. The case k = 3. Alternative n-cluster structure, N = 3n. In this case, similarity transformations (the interchange of rows and corresponding columns) reduce the coupling matrix $\mathbf{D}_{2n}$ to a similar matrix of the form

$$\mathbf{I}_2 \otimes \mathbf{D}_n \sim \mathbf{D}_{2n}; \quad \mathbf{D}_n = \|\mathbf{a}, \ldots, \mathbf{a}, \mathbf{b}\|, \quad \mathbf{a} = \mathrm{col}(0, 0, \ldots, 0), \quad \mathbf{b} = \mathrm{col}(-1, 0, \ldots, 0, 2)$$

By analogy with Theorem 2, there is a non-degenerate transformation

$$\mathbf{U} = \mathbf{S}\mathbf{V}, \quad \mathbf{S} = \mathbf{S}_0 \otimes \mathbf{I}_2$$

where $\mathbf{S}_0$ is the similarity transformation matrix for $\mathbf{D}_{2n}$, that brings Eq. (4.1) into the form

$$\dot{\mathbf{V}} = (\mathbf{I}_2 \otimes \mathbf{J}_{2n}(\xi) - \sigma_0^* \mathbf{I}_2 \otimes \mathbf{D}_n \otimes \mathbf{C})\mathbf{V}, \quad \mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2)^T \tag{4.3}$$

This equation, in turn, breaks down into a pair of identical equations of type (4.2) with the matrix $\mathbf{D}_n$ indicated. Applying the assumptions of the perturbation theory described above to each of these equations, we obtain the stability of a cluster structure with regular internal dynamics and a fairly small coupling parameter. In this case, $D = 3$.

Note that the roots of the cluster matrix $\mathbf{B}_s$

$$\lambda_j = 4\sin^2(\pi j/2n)$$

and the roots of the matrix $\mathbf{B}_s + \mathbf{D}_n$

$$\lambda_j^* = 4\sin^2(3\pi(2j+1) + (-1)^{j-1}\pi/(12n))$$

where $j = 0, 1, \ldots, n-1$, are related by the inequality $\lambda_j^* = \lambda_j$, i.e., the entire spectrum of the matrix $-(\mathbf{B}_s + \mathbf{D}_n)$ is located to the left of the spectrum of the matrix $-\mathbf{B}_s$.

Thus, in the case of the regular cluster attractors $A_s(n)$, the synchronization of three cluster rotators is stable at least at a low coupling level. Accordingly, the cluster structure shown in Fig. 1*b* is stable if all the cluster rotators except the first three are removed from it. In a chain of Froude pendulums with $k = 3$ and $n = 2$, we obtain a two-cluster structure by removing one cluster rotator (two elementary rotators) among the cluster rotators shown in Fig. 2.

The stability of cluster structures with regular dynamics is established in a similar manner in cases with $k > 3$.

4.2. Asymmetric cluster rotators and cluster structures associated with them

The stability conditions of cluster structures based on cluster rotators of type $R_a(n)$ consist of the conditions for the stability of the central cluster structure in a simple cell and the conditions for the stability of the synchronization of the simple cells themselves as structure-forming objects.

We will investigate the synchronization of a pair of inertially coupled cluster rotators of type $R_a(n)$. They are described by the system

$$\dot{\mathbf{X}} = \mathbf{G}(\mathbf{X}) - \sigma_0 \mathbf{B}_a \otimes \mathbf{C}\mathbf{X} + \sigma_0^* \mathbf{D}(-\mathbf{X} + \mathbf{Y}), \quad \dot{\mathbf{Y}} = \mathbf{G}(\mathbf{Y}) - \sigma_0 \mathbf{B}_a \otimes \mathbf{C}\mathbf{Y} + \sigma_0^* \mathbf{D}(\mathbf{X} - \mathbf{Y}) \tag{4.4}$$

$$\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T, \quad \mathbf{G}(\mathbf{X}) = (\mathbf{F}(\mathbf{X}_1), \mathbf{F}(\mathbf{X}_2), \ldots, \mathbf{F}(\mathbf{X}_n))^T$$

$$\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)^T, \quad \mathbf{G}(\mathbf{Y}) = (\mathbf{F}(\mathbf{Y}_1), \mathbf{F}(\mathbf{Y}_2), \ldots, \mathbf{F}(\mathbf{Y}_n))^T$$

$$\mathbf{D} = \mathbf{D}_n = \|\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b}\|, \quad \mathbf{a} = \mathrm{col}(0, \ldots, 0, 1), \quad \mathbf{b} = \mathrm{col}(0, \ldots, 0, 0)$$

The linearized system in differences between like variables of the cluster rotators has the form

$$\dot{\mathbf{U}} = (\mathbf{J}_{2n}(\xi) - \sigma_0^* \mathbf{D})\mathbf{U}$$

$$\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_n)^T, \quad \mathbf{U}_i = \mathbf{X}_i - \mathbf{Y}_i, \quad i = 1, 2, \ldots, n, \quad \mathbf{J}_{2n} = \mathbf{G}'(\xi) - \sigma_0 \mathbf{B}_\alpha \otimes \mathbf{C},$$

$$\xi \in A_a(n) \tag{4.5}$$

When $\sigma_0^* = 0$, Eq. (4.5) is an equation in variations for the equation of a cluster rotator of type $R_a(n)$ relative to the solution $\xi(t) \in A_s(n)$. If the cluster attractor $A_s(n)$ is regular, this ("generating") equation is stable. The perturbation norm satisfies the inequality

$$\left\|\sigma_0^* \mathbf{D}\mathbf{U}\right\| \le \sigma_0^* \|\mathbf{D}\| \cdot |\mathbf{U}| = \sigma_0^* D|\mathbf{U}|; \quad D = \|\mathbf{D}\|$$

In this case $D = 1$. As above, when the coupling parameter σ_0^* is sufficiently small, the leading exponent of the perturbed equation obeys the inequality $\chi(\mathbf{U}) \le \Lambda + D\sigma_0^* < 0$, where $\Lambda < 0$ is the leading exponent of the unperturbed equation. Thus, the trivial solution of Eq. (4.5) and the synchronization of the cluster rotators are stable.

We now assume that $\mathbf{X}_n = \mathbf{Y}_n$ (rigid coupling), $\mathbf{Y}_i = \mathbf{X}_{2n-i}$ ($i = 1, 2, \ldots, n$), and $\sigma_0^* = \sigma_0$. In this case, system (4.4) describes the simple cell depicted in Fig. 1*c*.

When structures based on simple cells are synthesized, the latter are joined to one another just like symmetric cluster rotators. The stability of the synchronization of simple cells as structure-forming objects and, as a consequence, the stability of the corresponding cluster structures is investigated in the same way as the case of symmetric cluster rotators.

5. A non-autonomous chain

We will investigate the dynamical properties of a non-autonomous system in the principal resonance zone by the method of averaging.

5.1. Transformation of the system to a standard form

Using the replacement of variables

$$\dot{\varphi}_i = \omega_0 + \mu F_i(\varphi_i, \psi, \xi), \quad F_i = \frac{\cos\varphi_i}{\omega_0} - \frac{A\cos\psi}{\omega_0} + \xi_i, \quad \eta_i = \varphi_i - \psi$$

where $\mu = C^{-1}$,[17] we reduce system (2.1) to the equivalent system of the form

$$\dot{\xi}_i = \mu\left(-F_i\frac{\partial F_i}{\partial\varphi_i} - F_i + \sigma(F_{i-1} - 2F_i + F_{i+1}) + \Delta\right), \quad \dot{\eta}_i = \mu F_i, \quad \dot{\psi} = \omega_0 \tag{5.1}$$

where $\gamma - \omega_0 = \mu\Delta$ is the frequency mismatch (principal resonance zone) and $\eta_i = \varphi_i - \psi$ are the phase mismatches. The parameters of the individual rotators lie in regions *2* and *3* (Fig. 5*a*), in which there are rotational limit cycles.

In the case when $\mu = C^{-1} << 1$, system (5.1) has the standard form of a system with one rapidly rotating phase ψ.[18]

5.2. The averaged system

Averaging system (5.1) over the fast phase and transforming the time $\mu\tau = \tau_n$, we obtain the averaged system

$$\dot{\eta}_i = \xi_i, \quad \dot{\xi}_i = -\xi_i - \frac{A}{2\omega_0^2}\sin\eta_i + \Delta + \sigma(\xi_{i-1} - 2\xi_i + \xi_{i+1}) \tag{5.2}$$

with boundary conditions

$$\xi_0 = \xi_1, \quad \xi_N = \xi_{N+1}$$

System (5.2), in turn, is reduced, by replacement of the time

$$\sqrt{\frac{A}{2\omega_0^2}}\tau_n = \tau_{nn}$$

to the single equation

$$\ddot{\eta}_i + \lambda_0\dot{\eta}_i + \sin\eta_i = \gamma_0 + \sigma_0(\dot{\eta}_{i-1} - 2\dot{\eta}_i + \dot{\eta}_{i+1}), \ i = 1, 2, \ldots, N, \ \eta_0 = \eta_1, \ \eta_N = \eta_{N+1} \tag{5.3}$$

Here

$$\lambda_0 = \sqrt{\frac{2\omega_0^2}{A}}, \quad \gamma_0 = \frac{2\omega_0^2\Delta}{A}, \quad \sigma_0 = \sqrt{\frac{2\omega_0^2}{A}}\sigma$$

The averaged system (5.3) has the same form as the system of equations of an autonomous chain. The partitioning of the (γ_0, λ_0) parameter plane into regions with qualitatively different dynamics for the "averaged" elementary rotator is generally the same as that shown in Fig. 5*a* with known structures of the phase trajectories (Fig. 5*b-d*). The difference is confined to the fact that regions *1* and *2* are separated by a certain band of width $\sim\mu$, which is "stretched" onto the Tricomi curve, rather than by the Tricomi curve itself. For parameters from this band, the phase space of the non-autonomous rotator contains attracting homoclinic structures and the chaotic limit sets of phase trajectories associated with them.[19] It follows from the principle of averaging that if L is a certain limit set of trajectories of the averaged autonomous system, $L \times S^1$ is the corresponding limit set of the non-autonomous system together with the stability conditions of L. On this basis the dynamical properties of the autonomous chain (5.3) are reformulated for the non-autonomous case. For example, region *1* in the (γ_0, λ_0) parameter plane is the region of global asymptotic stability of the limit cycle of the non-autonomous rotator (the synchronization region), region *3* is the region of existence of the stable two-dimensional torus T^2, region *2* is the region of existence of both of the limit sets that are realized depending on the initial conditions, and regions *2* and *3* are the regions of existence of the cluster attractors. The investigation of the stability of the cluster structures in a non-autonomous chain does not differ from the autonomous case.

Example 3. *Numerical experiment.* Cluster structures based on the cluster rotators $R_a(2)$ and $R_s(2)$ are synthesized using the method described in Section 1. The stability of all the cluster structures that exist in a chain of six elementary rotators was investigated.

An autonomous chain. System (3.1) was examined with the parameter values $\lambda = 0.07$, $\gamma = 0.28$ and $\sigma = 0.29$ for the case of $N = 6$. Fig. 6*a–c* shows a simple cell of cluster rotators of type $R_a(2)$. The cluster attractor $A_a(2)$ is a rotationally-oscillatory limit cycle. The middle rotator rotates, and the extreme rotators have an oscillatory regime. The oscillation amplitude is insignificant. The central cluster structure forms on the basis of a pair of simple cells. The cluster structure is stable.

A non-autonomous chain. System (3.1) was examined for the parameter values

$$\lambda = 0.07, \quad \gamma = 0.51, \quad \sigma = 0.079, \quad A = 0.28, \quad \omega_0 = 0.56$$

after adding a non-autonomous perturbation of the form $A\sin(\omega_0 t)$.

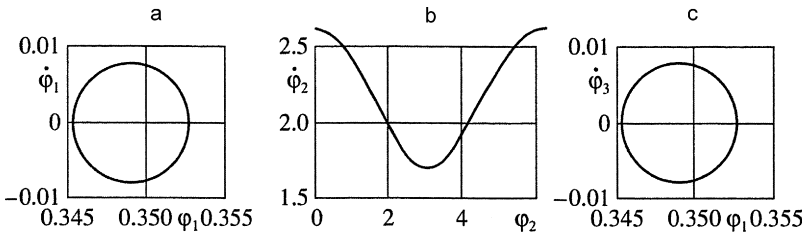

Fig. 6.

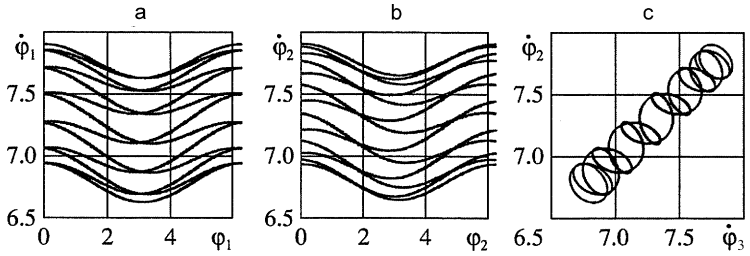

Fig. 7.

Fig. 7*a* and *b* depict the cluster attractor $A_s(2)$, which is the T^3 torus of the cluster rotator $R_s(2)$ in a projection onto the coordinate planes of the elementary rotators. To illustrate the absence of synchronization of the elementary rotators, Fig. 7*c* shows a projection of this attractor onto the plane of their like variables.

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References

1. Josić K. Invariant manifolds and synchronization of coupled dynamical systems. *Phys Rev Lett* 1998;**80**(14):3053–6.
2. Belykh VN, Belykh IV, Hasler M. Hierarchy and stability of partially synchronous oscillations of diffusely coupled dynamical systems. *Phys Rev E* 2000;**62**(5):6332–45.
3. Belykh VN, Belykh IV, Mosekilde E. Cluster synchronization modes in an ensemble of coupled chaotic oscillators. *Phys Rev E* 2001;**63**(3), 036216-1–4.
4. Verichev NN, Verichev SN, Wiercigroch M. Physical interpretation and theory of existence of cluster structures in lattices of dynamical systems. *Chaos, Solitons and Fractals* 2007;**34**(4):1082–104.
5. Afraimovich VS, Verichev NN, Rabinovich MI. Stochastic synchronization of oscillations in dissipative systems. *Izv Vuzov Radiofizika* 1986;**29**(9):1060–160.
6. Watanabe S, van der Zant HSJ, Strogatz SH, Orlando TP. Dynamics of circular arrays of Josephson junctions and the discrete sine-Gordon equation. *Physica D* 1996;**97**(4):429–70.
7. Afraimovich VS, Nekorkin VI, Osipov GV, Shalfeev VD. *Stability, Structures and Chaos in Nonlinear Synchronization Networks*. Singapore: World Scientific; 1994.
8. Barone A, Paterno G. *Physics and Applications of the Josephson Effect*. New York: Wiley; 1982.
9. Bautin VN, Leontovich YeA. *Methods and Means for the Qualitative Investigation of Dynamical Systems in a Plane*. Moscow: Nauka; 1990.
10. Akimov VN, Belyustina LN, Belykh VN, et al. *Phase Synchronization Systems*. Moscow: Radio i Svyaz'; 1982.
11. Pliss VA. *Integral Sets of Periodic Systems of Differential Equations*. Moscow: Nauka; 1977.
12. Magnus K. *Schwingungen*. Stuttgart: Teubner; 2005.
13. Likharev KK. *Introduction to the Dynamics of Josephson Junctions*. Moscow: Nauka; 1985.
14. Lancaster P. *The Theory of Matrices with Applications*. New York: Academic Press; 1985.
15. Demidovich BP. *Lectures on the Mathematical Theory of Stability*. Moscow: Nauka; 1967.
16. Bylov BF, Vinograd RE, Grobman DM, Nemytskii VV. *Theory of Lyapunov Exponents and its Application to Stability Problems*. Moscow: Nauka; 1966.
17. Verichev NN. Investigation of systems with Josephson junctions using the rapidly rotating phase method. *Radiotekh Elektron* 1986;**31**(11):2267–74.
18. Volosov VM, Morgunov BI. *The Method of Averaging in the Theory of Non-linear Oscillatory Systems*. Moscow: Izd MGU; 1971.
19. Arnold VI. *Additional Chapters in the Theory of Ordinary Differential Equations*. Moscow: Nauka; 1978.

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